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The Statistics of Curie–Weiss Models

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Let S_n denote the random total magnetization of an n-site Curie–Weiss model, a collection of n (spin) random variables with an equal interaction of strength $1/n$ between each pair of spins. The asymptotic behavior for large n of the probability distribution of S_n is analyzed and related to the well-known (mean-field) thermodynamic properties of these models. One particular result is that at a type-k critical point $(S_n - nm)/n^{1-1/2k}$ has a limiting distribution with density proportional to $\exp[-\lambda s^{2k}/(2k)!]$, where m is the mean magnetization per site and λ is a positive critical parameter with a universal upper bound. Another result describes the asymptotic behavior relevant to metastability.

KEY WORDS: Block spin; renormalization group; mean-field; Curie–Weiss.

1. INTRODUCTION

The classical Curie–Weiss theory of magnetism occupies a central place in the physical literature. Based on the device of a self-consistent (or mean) field, the theory allows one to readily study the behavior of thermodynamic quantities such as specific heat, isothermal susceptibility, and magnetization in the neighborhood of the critical point. Unfortunately, the predictions of this classical theory do not completely agree with experiment, and so other theories, like nearest neighbor Ising models, must be considered. However, because of its relative simplicity and the qualitative correctness of at least some of its predictions (e.g., it works well away from the critical point), the Curie–Weiss theory has been historically important.[1,16]

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A relatively recent approach to critical behavior is the renormalization group method.[12,19] An underlying idea of this method is that a critical point for (say) a lattice model of magnetism can be analyzed probabilistically by analyzing the asymptotics of block spins; i.e., sums of the spin random variables in the model.[9,10] In particular, the validity or nonvalidity of the central limit theorem should be related to the noncriticality or criticality of phase, while the parameters appearing in limiting distributions of suitably scaled block spins should be directly related to critical exponents in the model.

The purpose of this paper is to present, without proofs, block-spin limit theorems and related results for one such model, the Curie–Weiss model. As shown by Kac,[11] the thermodynamics of this model coincides with that of the classical Curie–Weiss theory of magnetism. Just as the classical Curie–Weiss theory has been helpful in one approach to the study of critical behavior, it is hoped that our probabilistic results on Curie–Weiss models will be helpful in analyzing analogous phenomena in more realistic systems. The Curie–Weiss model is often considered to exhibit uninteresting statistical behavior, since the fluctuations of block spins are usually thought to be normally distributed. One surprise of the present research is that this is not at all the case, provided that the model is properly viewed. Then the rich probabilistic structure of the Curie–Weiss model emerges.

Some of the results presented here appeared first in Ref. 5, which can be consulted for detailed proofs. For the proofs of more recent results and for extensions to other models, a full-length study is planned. A discussion of relevant background material more complete than could be given here is included in Ref. 6.

2. DEFINITIONS AND PRELIMINARY RESULTS

Let ρ be a nondegenerate Borel probability measure on $\mathbb{R}^1$ satisfying

$$\int \exp(x^2/2)\, d\rho(x) < \infty \tag{1}$$

The Curie–Weiss model is defined as the triangular array of spin random variables $\{X_j^{(n)}(\rho): j = 1,\dots, n\}$ $(n = 1, 2,\dots)$ with joint distribution

$$dP_n(\rho; x_1,\dots, x_n) = \frac{1}{Z_n(\rho)} \exp\left[\frac{(x_1 + \cdots + x_n)^2}{2n}\right] \prod_{j=1}^{n} d\rho(x_j) \tag{2}$$

where

$$Z_n(\rho) = \int_{\mathbb{R}^n} \exp\left[\frac{(x_1 + \cdots + x_n)^2}{2n}\right] \prod_{j=1}^{n} d\rho(x_j) \tag{3}$$

Here j denotes site location in a finite lattice of n sites. In our formulation, all thermodynamic parameters are introduced via the ρ dependence in (2). For example, inverse temperature β and external field h are brought in by replacing $d\rho$ by

$$d\rho_{\beta,h}(x) = \exp(\sqrt{\beta}\, h)_{,} d\rho(x/\sqrt{\beta}) \tag{4}$$

There are two natural choices of block spin variable, distinguished by whether the thermodynamic limit is taken together with or prior to the long-range order limit (we use terminology adapted from Refs. 8 and 14):

the long-order block spin

$$S_n{}^l(\rho) = \sum_{j=1}^{n} X_j^{(n)}(\rho) \tag{5}$$

and the short-order block spin

$$S_n{}^s(\rho) = \sum_{j=1}^{n} X_j^{(\infty)}(\rho) \tag{6}$$

Here, $\{X_j^{(\infty)}(\rho) : j = 1, 2, ...\}$ are random variables with joint distribution $dP_\infty(\rho; x_1, x_2, ...)$ on $\mathbb{R}^\infty$, the finite-dimensional distributions of which are defined to be the weak limits, as $n \to \infty$, of the corresponding finite-dimensional distributions of dP_n in (2). The formula for dP_∞ will be given in Section 5. From this formula, it will be clear that the statistics of $S_n{}^s(\rho)$ are trivial (i.e., asymptotically normally distributed fluctuations regardless of ρ). By contrast, the possible kinds of fluctuations of $S_n{}^l(\rho)$ will be seen in Section 3 to be quite varied and to depend on ρ in an interesting way. From now on, the ρ in the notation for $Z_n(\rho)$, $X_j^{(n)}(\rho)$, $X_j{}^\infty(\rho)$, $S_n{}^l(\rho)$, $S_n{}^s(\rho)$ will be dropped, provided there is no danger of confusion.

The specific free energy $f(\rho)$ is defined by the limit

$$f(\rho) = -\lim_{n\to\infty} \frac{1}{n} \ln Z_n \tag{7}$$

The following variational formula for $f(\rho)$ is easily derived (Ref. 11; Ref. 18, p. 100). Here and below all integrals extend over $\mathbb{R}^1$ unless otherwise noted.

Proposition. Define

$$G(z) = G_\rho(z) \equiv \tfrac{1}{2}z^2 - \ln \int \exp(zx)\, d\rho(x), \qquad z \text{ real} \tag{8}$$

Then

$$f(\rho) = \inf_{z \text{ real}} G_\rho(z) \tag{9}$$

$G(z)$ is real analytic and $G(z) \to \infty$ as $|z| \to \infty$, so that G has only a finite number of global minima.

Because of (9), topological properties of G correspond to thermodynamic properties of the Curie–Weiss model. Thus, multiple global minima of G correspond to coexisting phases, the magnetizations of which are the various locations of the minima; nonquadratic global minima correspond to critical phases, local minima to metastable phases, and points of inflection to spinodal points. As we show in the next section, these topological properties of G also dictate the asymptotic behavior of $S_n{}^l$.

Given a positive integer α, α distinct real numbers $m_1,\ldots, m_\alpha$, and α positive integers $k_1,\ldots, k_\alpha$, we say that the vector $(m_1, k_1;\ldots; m_\alpha, k_\alpha)$ is *admissible*. We write

$$\rho \sim (m_1, k_1;\ldots; m_\alpha, k_\alpha)$$

if the set of global minima of G_ρ is $\{m_1,\ldots, m_\alpha\}$ and for each $i = 1,\ldots, \alpha$,

$$G_\rho(z) = G_\rho(m_i) + \frac{\lambda_i(z - m_i)^{2k_i}}{(2k_i)!} + O[(z - m_i)^{2k_i}], \qquad \text{as } z \to m_i \quad (10)$$

where λ_i is a positive real number. We call $k(m_i) \equiv k_i$ the *type* and $\lambda(m_i) \equiv \lambda_i$ the *strength* of the minimum m_i; these definitions also make sense for a local minimum. The *maximal type* is defined as the largest of the k_i. The measure ρ is said to be *pure* if G_ρ has a unique global minimum and *semipure* if it has a unique global minimum of maximal type; such measures are said to be *critical* if the type (or maximal type) exceeds one. A pure (or semipure) measure is said to be *centered* at m, the location of the unique global minimum (of maximal type); its type is the type $k(m)$ of m.

We note that if ρ is semipure and centered at m_0, then $m = m_0$ is a solution of $G_\rho'(m) = 0$, or equivalently,

$$m = \int x \exp(mx)\, d\rho(x) \Big/ \int \exp(mx)\, d\rho(x) \qquad (11)$$

This is just the "self-consistent field equation," which forms the basis of the classical Curie–Weiss theory (Ref. 1, p. 9). For a general ρ, the solutions of (11) give all the local maxima and minima of G_ρ; thus (11) has a unique solution m_0 only when ρ is pure, centered at m_0, and G_ρ has no other local minima. It is an intriguing fact that although G_ρ can only have a finite number of global minima, ρ's exist for which G_ρ has infinitely many local minima.

Reformulation of Condition of Purity

Define the moments

$$\mu_j(\rho) = \int x^j\, d\rho(x), \qquad j = 0, 1, 2,\ldots \qquad (12)$$

and let $\bar{\mu}_j$ denote the moments of a standard normal random variable:

$$\bar{\mu}_j = (2\pi)^{-1/2} \int x^j \exp(-x^2/2)\, dx, \qquad j = 0, 1, 2,... \tag{13}$$

One can then easily see that ρ is pure, centered at zero, and of type k if and only if

$$\int \exp(zx)\, d\rho(x) < \exp(z^2/2), \qquad \text{for all } z \text{ real}, \quad z \neq 0 \tag{14}$$

and

$$\bar{\mu}_j - \mu_j(\rho) = \begin{cases} 0, & j = 0, 1, 2,..., 2k - 1 \\ \lambda > 0, & j = 2k \end{cases}$$

As examples, we have

$$\rho = \tfrac{1}{2}[\delta(x - 1) + \delta(x + 1)] \qquad \text{for} \quad k = 2 \tag{15}$$

$$\rho = \tfrac{2}{3}\delta(x) + \tfrac{1}{6}[\delta(x - \sqrt{3}) + \delta(x + \sqrt{3})] \qquad \text{for} \quad k = 3 \tag{16}$$

The latter measure was used in Ref. 2 to analyze the tricritical point of liquid helium.

Given any admissible vector $(m_1, k_1;...; m_\alpha, k_\alpha)$, Theorem 6 in the next section guarantees the existence of a measure ρ satisfying $\rho \sim (m_1, k_1;...; m_\alpha, k_\alpha)$. As a corollary of this, one obtains pure measures ρ with $\rho \sim (0, k)$ for each $k = 2, 3, 4,...$.

3. LIMIT THEOREMS FOR LONG-ORDER BLOCK SPINS; EXISTENCE OF CRITICAL MEASURES

Throughout this section, we write S_n for $S_n{}^l$. Our first theorem is a law of large numbers-type result for S_n. The second theorem corresponds to the classical central limit theorem and the classical stable law limit theorems for sums of independent, identically distributed random variables. But we emphasize that the limiting measures appearing in these results are not the classical stable distributions. Theorem 3 describes the infinitesimal neighborhood of type-k critical measures ρ by studying the asymptotics of $S_n(\rho_n)$ as $n \to \infty$, where $\{\rho_n: n = 1, 2,...\}$ are measures that tend to ρ as $n \to \infty$. Theorem 4 yields the asymptotics of multiple block spins; i.e., the limit $r \to \infty$ in a Curie–Weiss model with $n = qr$ sites, which we think of as consisting of q blocks of r sites each. Theorem 5 is a conditional version of Theorem 2, extending the latter to the case where ρ is an arbitrary measure (not necessarily semipure) and m is either an arbitrary minimum of G_ρ (not necessarily global or of maximal type) or a point of inflection of G_ρ. Thus Theorem 5 should be viewed as describing the critical points (and spinodal points) of metastable phases. Theorems 3 and 4 may be similarly extended to the metastable context by conditioning. Finally, Theorem 6 guarantees the existence of measures ρ such that G_ρ has prescribed global minima of various types. Most of these results extend naturally to Curie–Weiss rotator models, which are Curie–Weiss models with a spherically symmetric, single-spin measure ρ on $\mathbb{R}^d$, $d \in \{2, 3, \ldots\}$. On the other hand, to extend all our theorems to the non-spherically symmetric case is an interesting open problem, which will be discussed in Section 6. We are also lacking a result which would generalize Theorem 6 by guaranteeing the existence of measures ρ such that G_ρ has prescribed *local* minima of various types. Special cases of Theorems 2 and 3 appeared in Refs. 15 and 3. The key ingredient in the proofs of Theorems 1–5 and 8 is the simple fact that if W is a unit normal random variable independent of $S_n(\rho)$, then $S_n(\rho)/n + W/\sqrt{n}$ has distribution proportional to $\exp[-nG_\rho(x)]\,dx$.

Before we state the results, we comment on their physical content. The existence of non-Gaussian limits with scalings $n^{1-1/2k}$ for $k \in \{2, 3, 4, \ldots\}$, as in Theorem 2, is consistent with Kadanoff's picture of critical behavior as a mass scale phenomenon involving strongly correlated individual spins (Ref. 12, §I.E). Strong correlations over large distances are what differentiate a system near the critical point from a system far from the critical point. These strong correlations, in turn, modify the asymptotic mass scale behavior of the system away from the usual Gaussian limit. In this sense, the Curie–Weiss model (2) with a critical ρ may be said to have a critical inverse temperature at $\beta_{\text{crit}} = 1$. We justify this terminology further by singling out a class of measures for which the corresponding block spins have a non-Gaussian limit if and only if the inverse temperature β equals 1; for $\beta \neq 1$, a Gaussian limit arises. Consider the class of symmetric probability measures ρ with $\mu_2(\rho) = 1$, $\int \exp(\gamma x^2)\,d\rho(x) < \infty$ for all $\gamma > 0$ and such that ρ satisfies the GHS inequality (see discussion in Ref. 5). Such a ρ may be shown to be pure, centered at 0, and of type $k \geqslant 2$, and so by Theorem 2 the corresponding block spins have a non-Gaussian limit. Now consider the measure $d\rho_\beta(x) \equiv d\rho(x/\sqrt{\beta})$ [see discussion before (4)]. For $\beta < 1$, $d\rho_\beta$ is pure, centered at 0, but of type $k = 1$ (corresponding to the existence of a single, noncritical phase in the high-temperature region), and a Gaussian block-spin limit arises. For $\beta > 1$, G_{ρ_β} has two global minima at $m = \pm m(\beta)$, $m(\beta) > 0$, each of type $k = 1$ (corresponding to the existence of two noncritical phases in the low-temperature region). Now an application of Theorem 5 shows that, conditional upon the average spin being near one of these two values, a Gaussian limit arises. See Ref. 7 for a discussion of the central limit theorem away from the critical temperature for more general models.

Given $\rho \sim (m_1, k_1; \ldots; m_\alpha, k_\alpha)$, we define the measure

$$d\tau_\rho(m) = \sum_{j=1}^{\alpha} b_j \, \delta(m - m_j) \tag{17}$$

where $b_j = \bar{b}_j / \sum \bar{b}_j$ and

$$\bar{b}_j = \begin{cases} [\lambda(m_j)]^{-1/2k_j} & \text{if } k_j \text{ is of maximal type} \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

If certain random variables or vectors $\{Y_n : n = 1, 2, \ldots\}$ converge weakly, as $n \to \infty$, to a random variable with distribution $d\nu(x)/\int d\nu(x)$, we write $Y_n \to d\nu(x)$. Given random variables $\{Y_n, W_n : n = 1, 2, \ldots\}$, a Borel subset A of $\mathbb{R}$, and a measure $d\nu(x)$, the notation

$$Y_n | \{W_n \in A\} \to d\nu(x) \tag{19}$$

means that as $n \to \infty$, the conditional distribution of Y_n, given that W_n is in A, converges weakly to the distribution $d\nu(x)/\int d\nu(x)$.

Theorem 1. If $\rho \sim (m_1, k_1; \ldots; m_\alpha, k_\alpha)$, then

$$S_n/n \to d\tau_\rho \tag{20}$$

Theorem 2. If ρ is semipure, centered at m, and of type k, then

$$(S_n - nm)/n^{1-1/2k} \to \begin{cases} \exp(-x^2/2\sigma^2)\, dx & \text{if } k = 1 \\ \exp[-\lambda(m) x^{2k}/(2k)!]\, dx & \text{if } k \geqslant 2 \end{cases} \tag{21}$$

Here $\sigma^2 \equiv [\lambda(m)]^{-1} - 1$ can be shown to be positive.

Theorem 3. If ρ is pure, centered at m, and of type k, then for any real $\lambda_1, \ldots, \lambda_{2k-1}$, measures ρ_n can be chosen so that $\rho_n \to \rho$ and

$$\frac{S_n(\rho_n) - nm}{n^{1-1/2k}} \to \exp\left[-\lambda(m)\frac{x^{2k}}{(2k)!} - \sum_{j=1}^{2k-1} \lambda_j \frac{x^j}{j!}\right] dx \tag{22}$$

Theorem 4. Assume that ρ is semipure, centered at m, and of type k, and denote by $f_k(x)\, dx$ the limiting measure in (21). For fixed q, let $n = qr$ in (2) and define

$$S_{n,1} = \sum_{j=1}^{r} X_j^{(n)}, \qquad S_{n,2} = \sum_{j=r+1}^{2r} X_j^{(n)}, \ldots, \qquad S_{n,q} = \sum_{j=(q-1)r+1}^{qr} X_j^{(n)} \tag{23}$$

Then the random vector $\mathbf{S}_n = (S_{n,1},..., S_{n,q})$ has the following asymptotics as $r \to \infty$ [where $\mathbf{m} \equiv (m,..., m)$]:

$$(\mathbf{S}_n - r\mathbf{m})/r^{1-1/2k} \to \begin{cases} \exp\left[-\sum_{i,j=1}^{q} (x_i - x_j)^2/2q\right] \prod_{i=1}^{q} f_1(x_i)\, dx_i \\ \quad \text{if} \quad k = 1 \\ \left[\prod_{i=1}^{q} f_k(x_i)\right] d\nu_q(x_1,..., x_q) \qquad \text{if} \quad k \geqslant 2 \end{cases} \tag{24}$$

where $d\nu_q$ is (one-dimensional) Lebesgue measure supported on $\{x_1 = x_2 = \cdots = x_q\}$.

Theorem 5. (a) Assume that m is either a nonunique global minimum or a local minimum of G_ρ. Then there exists $\epsilon' > 0$ such that for all $0 < \epsilon < \epsilon'$

$$\frac{S_n}{n}\bigg|\left\{\frac{S_n}{n} \in [m - \epsilon, m + \epsilon]\right\} \to \delta(x - m) \tag{25}$$

$$\frac{S_n - nm}{n^{1-1/2k}}\bigg|\left\{\frac{S_n}{n} \in [m - \epsilon, m + \epsilon]\right\} \to \begin{cases} \exp(-x^2/2\sigma^2)\, dx & \text{if} \quad k = 1 \\ \exp[-\lambda x^{2k}/(2k)!]\, dx & \text{if} \quad k \geqslant 2 \end{cases} \tag{26}$$

where k is the type and λ the strength of m, and σ^2 is as in Theorem 2. (b) Assume that m is a point of inflection of G_ρ and that

$$G_\rho(z) = G_\rho(m) + \frac{\lambda(z - m)^{2k+1}}{(2k + 1)!} + o[(z - m)^{2k+1}] \qquad \text{as } z \to m \tag{27}$$

for some $k \in \{1, 2,...\}$ and λ real. Assume that $\lambda > 0$ (a similar result holds for $\lambda < 0$). Then there exists $\epsilon' > 0$ such that for all $0 < \epsilon < \epsilon'$

$$\frac{S_n}{n}\bigg|\left\{\frac{S_n}{n} \in [m, m + \epsilon]\right\} \to \delta(x - m) \tag{28}$$

$$\frac{S_n - nm}{n^{1-1/(2k+1)}}\bigg|\left\{\frac{S_n}{n} \in [m, m + \epsilon]\right\} \to d\nu(x) \equiv \begin{cases} \exp[-\lambda x^{2k+1}/(2k + 1)!]\, dx & \text{for} \quad x \geqslant 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{29}$$

We end this section with a theorem on the existence of measures with prescribed global minima of various types. The proof follows from results in the theory of moments.[13]

Theorem 6. Given any admissible vector $(m_1, k_1;...; m_\alpha, k_\alpha)$ with $k \equiv k_1 + \cdots + k_\alpha \geqslant 2$, there exists a unique probability measure $\bar{\rho}$ supported on k points such that $\bar{\rho} \sim (m_1, k_1; \ldots; m_\alpha, k_\alpha)$. If $\rho \neq \bar{\rho}$ and also $\rho \sim (m_1, k_1; \ldots; m_\alpha, k_\alpha)$, then ρ is not supported on fewer than $k + 1$ points and

$$\lambda_\rho(m_i) < \lambda_{\bar{\rho}}(m_i) \qquad \text{for each } i = 1, \ldots, \alpha \tag{30}$$

In particular, for each m and for each $k \in \{2, 3, \ldots\}$, there exists a unique probability measure supported on k points which is pure, centered at m, and of type k. The strength of the minimum at m is $k!$ and (30) holds, so that if ρ is any other pure measure, centered at m, and of type k, then

$$\lambda_\rho(m) < k! \tag{31}$$

Explicit formulas are known for the probability measures of type $k \geqslant 2$, the existence of which is stated in the second half of the theorem (see Ref. 17, §3.8, §3.12; Ref. 5, Theorem 2.3). When $m = 0$, the support of this measure is the zero set of the kth Hermite polynomial. For $k = 2, 3$, the measures are given by (15)–(16). In the analogous theorem for rotator Curie–Weiss models, the minimally supported, spherically symmetric measure which is of type k at the origin has support on $k/2$ spherical shells, the origin counting as a half-shell for k odd. In particular, for $k = 2, 3$, these measures (on $\mathbb{R}^d$) are

$$\delta(|\mathbf{x}| - \sqrt{d}) \qquad \text{for} \quad k = 2 \tag{32}$$

$$\frac{2}{d+2}\,\delta(|\mathbf{x}|) + \frac{d}{d+2}\,\delta(|\mathbf{x}| - (d+2)^{1/2}) \qquad \text{for} \quad k = 3 \tag{33}$$

4. OTHER RESULTS

In this section, we first discuss the renormalization group equation for the Curie–Weiss model and then point out the connection between the limit results of Section 3 and critical exponents in the Curie–Weiss theory of magnetism.

The renormalization group equation is derived by choosing a real number $\theta \in [1, 2]$ and supposing that $S_n{}^l/n^{\theta/2}$ has an unnormalized probability density φ_n which tends to φ_∞ as $n \to \infty$. It is easily determined that

$$\varphi_{2n}(z) = \left(\frac{\gamma_n}{\pi}\right)^{1/2} \int \exp(-\gamma_n u^2)\, \varphi_n\!\left(\frac{z}{\sqrt{c}} + u\right) \varphi_n\!\left(\frac{z}{\sqrt{c}} - u\right) du \tag{34}$$

with $c = 2^{2-\theta}$ and $\gamma_n = (\text{const})n^{\theta-1}$. If $1 < \theta < 2$ (so that $2 > c > 1$) and we take $n \to \infty$, then (34) goes over formally to the fixed point equation

$$\varphi_\infty(z) = [\varphi_\infty(z/\sqrt{c})]^2 \tag{35}$$

The latter has solutions, $\exp(-\lambda|z|^p)$ for $p = 2/\log_2 c \in (2, \infty)$. It is noteworthy that only $p = 2k$ actually arises in the rigorous Curie–Weiss limit theorems of Section 3 and that even then there is an upper bound on λ [cf.

(30)]. If γ_n in (34) is replaced by 1, then we obtain the fixed point equation of the hierarchical model.[8]

We now turn to the connection between critical exponents and block-spin limit theorems for the Curie–Weiss model. Let ρ_c be pure, centered at zero, and of type k. A critical point at $\rho = \rho_c$ is analyzed thermodynamically by describing the behavior of $f(\rho)$ as $\rho \to \rho_c$ along various paths (in the thermodynamic parameter space); critical exponents are defined by the leading order behavior. For example, define $d\rho_h(x) \equiv \exp(hx)\, d\rho_c(x)$. Then $\rho_h \to \rho_c$ as $h \to 0$ and the critical exponent δ is defined by

$$f(\rho_h) = f(\rho_c) - C|h|^{1+1/\delta} + \text{higher order terms} \qquad \text{as } h \to 0 \tag{36}$$

Here, $C = C(\rho_c)$ is a positive constant. The following theorem, which puts a strong restriction on the possible values of δ, is representative of a number of standard results of this sort. Analogous results, which would restrict the possible values of critical exponents for more realistic models, would be most worthwhile.

Theorem 7. Let ρ_c and δ be as above. Then

$$\delta = 2k - 1 \tag{37}$$

In addition, one can show that the universal bounds of Theorem 6 [expressed in (30)–(31)] imply universal bounds on certain critical parameters associated with an arbitrary measure ρ_c (as in Theorem 7) in terms of the critical parameters associated with the minimally supported measure $\bar{\rho}$. For example, (31) yields such a lower bound on the constant C appearing in (36):

$$C(\rho_c) = \frac{2k-1}{2k}\left(\frac{(2k-1)!}{\lambda_c}\right)^{1/(2k-1)} \geqslant \frac{2k-1}{2k}\left(\frac{(2k-1)!}{k!}\right)^{1/(2k-1)} \tag{38}$$

where $\lambda_c = \lambda_{\rho_c}(0)$ is as in (31).

Analogous results for more realistic models would be attractive.

5. SHORT-ORDER BLOCK SPINS

We first describe the thermodynamic limit of the Curie–Weiss spins $\{X_j^{(n)}\}$. Results of this type were apparently first derived in Ref. 4 (by different methods).

Theorem 8. In the sense of weak convergence of finite-dimensional joint distributions,

$$\{X_j^{(n)}: j = 1,\ldots, n\} \to \{X_j^{(\infty)}: j = 1, 2,\ldots\} \tag{39}$$

where the joint distribution $dP_\infty(\rho; x_1, x_2,\ldots)$ of $\{X_j^{(\infty)}\}$ is given by

$$dP_\infty(\rho; x_1, x_2,\ldots\} = \int \left\{\prod_{j=1}^{\infty}\left[\exp(mx_j)\, d\rho(x_j) \Big/ \int \exp(mx)\, d\rho(x)\right]\right\} d\tau_\rho(m) \tag{40}$$

The measure τ_ρ is defined in (17). In particular, if ρ is semipure and centered at m_0, then $\{X_j^{(\infty)}\}$ is a set of independent random variables with common distribution $\exp(m_0 x)\, d\rho(x)/\int \exp(m_0 x)\, d\rho(x)$. If ρ is not semipure, then dP_∞ is a (finite) convex combination of such infinite product measures.

The (uninteresting) asymptotics for short-order block spins S_n^s now follow by the classical law of large numbers and central limit theorem. The result (42) can be extended to the nonsemipure case by conditioning (as in Theorem 5), with m one of the competing global minima of G_ρ of maximal type (but *not* with m one of the other global minima or local minima not appearing in τ_ρ); a Gaussian limit still arises.

Theorem 9. We have

$$S_n^s/n \to d\tau_\rho \tag{41}$$

If ρ is semipure and centered at m, then

$$(S_n^s - nm)/n^{1/2} \to \exp(-x^2/2\bar{\sigma}^2)\, dx \tag{42}$$

where $\bar{\sigma}^2 = \sigma^2/(\sigma^2 + 1)$; σ^2 is defined in Theorem 2.

Comparing this result with Theorems 1–2, we see that while the law of large numbers-type limit is the same for both long-order and short-order block spins, fluctuations about the mean (in the semipure case) are strikingly different. Even for a noncritical measure ($k = 1$), the right-hand sides of (21) and (42) differ.

6. STRANGE CRITICAL POINTS FOR VECTOR MODELS

As mentioned briefly above, most of our results have natural extensions to spherically symmetric Curie–Weiss rotator models. No unexpected phenomena occur in such models. In this section we discuss a specific example [see (47)] in order to point out some of the peculiarities which may occur in the absence of spherical symmetry. The example is based on a suggestion of Loren Pitt.

We suppose that ρ is a probability measure on $\mathbb{R}^d$ satisfying (1) (where x^2 denotes the square of the Euclidean length of $x \in \mathbb{R}^d$) and $\{X_j^{(n)}\}$ are d-dimensional random vectors with joint distribution given by (2) and (3) [with the integration in (3) over $(\mathbb{R}^d)^n$ rather than $\mathbb{R}^n$]. We define $f = f(\rho)$ and note that

$$f = \inf_{z \in \mathbb{R}^d} G_\rho(z) \tag{43}$$

where

$$G(z) = G_\rho(z) = \tfrac{1}{2} z^2 - \ln \int_{\mathbb{R}^d} \exp(zx)\, d\rho(x), \qquad z \in \mathbb{R}^d \tag{44}$$

and zx denotes the usual inner product in $\mathbb{R}^d$.

If $x = 0$ is the unique global minimum of $G(x)$ and ρ is spherically symmetric, then $G(x) = a(x^2)^k + O(x^2)^{k+1}$ for some $a > 0$ and $k \in \{1, 2, 3,...\}$. In this situation, $S_n = S_n(\rho) \equiv X_1^{(n)} + \cdots + X_n^{(n)}$ behaves asymptotically as in Theorem 2, with $S_n/n^{1-1/2k} \to \exp[-a(x^2)^k]\, dx$ on $\mathbb{R}^d$ (for $k \geqslant 2$).

To simplify notation in the absence of spherical symmetry, we let $d = 2$ and denote the components of S_n and x by $S_n = (U_n, V_n)$, $x = (u, v)$. One can clearly arrange to have a critical point in which the u and v components are critical of different type [e.g., by letting $\rho(x) = \rho_1(u)\rho_2(v)$ so that U_n and V_n are independent] with

$$G(x) = G(u, v) = a_1 u^{2k_1} + a_2 v^{2k_2} + O(|u|^{2k_1+1} + |v|^{2k_2+1}) \qquad (45)$$

In this situation, one must use a (diagonal) matrix scaling to analyze the asymptotics of S_n (say, in the case $1 < k_1 < k_2$):

$$(U_n/n^{1-1/2k_1}, V_n/n^{1-1/2k_2}) \to \exp(-a_1 u^{2k_1} - a_2 u^{2k_2})\, du\, dv \qquad (46)$$

The critical point of the next example is considerably stranger than that of (45) and the asymptotics of S_n are considerably more peculiar than those of (46).

We define $\rho(x)$ on $\mathbb{R}^2$ as

$$d\rho(u, v) = \tfrac{1}{3}\delta(u)\delta(v) + \tfrac{1}{6}\delta(u - \sqrt{3})\delta(v) + \tfrac{1}{6}\delta(u + \sqrt{3})\delta(v) + \tfrac{1}{6}\delta(u)\delta(v - \sqrt{3}) + \tfrac{1}{6}\delta(u)\delta(v + \sqrt{3}) \qquad (47)$$

Note that the marginal distribution of u (or v) from (47) is just the tricritical measure, (16). Straightforward calculations show that $G(x) = G_\rho(x)$ has a unique global minimum at the origin and that

$$G(u, v) = u^2v^2/4 + (u^6 + v^6)/120 - (u^4v^2 + u^2v^4)/16 + O(u^2 + v^2)^4 \qquad (48)$$

The minimum is thus of type 3 (i.e., degree 6) along the u, v axes and of type 2 along all other rays. It is easily shown that $S_n/n^{1-1/2k} \to \delta(u)\delta(v)$ for $k = 3$ and thus it is natural to try to obtain a nondegenerate limit with $k = 2$; this almost works. Formally, $S_n/n^{3/4} \to \exp(-u^2v^2/4)\, du\, dv$; unfortunately, $\exp(-u^2v^2/4)$ is nonintegrable (its integral over $\mathbb{R}^2$ is logarithmically divergent). To obtain a rigorous limit, we proceed, analogously to Theorem 3, by choosing an n-dependent temperature [according to the prescription (4)] which approaches the critical temperature from above as $n \to \infty$.

Theorem 10. Let $\beta_n = 1 - b/n^{1/2}$ with $b > 0$ and let $d\rho_n(x) = d\rho(x/\sqrt{\beta_n})$ with ρ given by (47); then

$$S_n(\rho_n)/n^{3/4} \to \exp[-u^2v^2/4 - b(u^2 + v^2)/2]\, du\, dv \qquad (49)$$

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